Instability Analysis of Cylindrical Stellar Object in Brans-Dicke Gravity

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Abstract

This paper investigates instability ranges of a cylindrically symmetric collapsing cosmic filamentary structure in Brans-Dicke theory of gravity. For this purpose, we use perturbation approach in the modified field equations as well as dynamical equations and construct a collapse equation. The collapse equation with adiabatic index (Γ) is used to explore the instability ranges of both isotropic as well as anisotropic fluid in Newtonian and post-Newtonian approximations. It turns out that the instability ranges depend on the dynamical variables of collapsing filaments. We conclude that the system always remains unstable for $0 < \Gamma < 1$ while $\Gamma > 1$ provides instability only for the special case.

Keywords: Brans-Dicke theory; Instability; Newtonian and post-Newtonian regimes.

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1 Introduction

Dark energy and gravitational collapse are the most fascinated and interesting phenomena of cosmology as well as gravitational physics. Number of astronomical observations such as Supernova type I, Sloan Digital Sky Survey, large scale-structure, Wilkinson Microwave Anisotropy Probe, galactic cluster emission of X-rays and weak lensing describe accelerated behavior of the expanding universe [1]. It is suggested that a mysterious type of energy known as dark energy is responsible for this accelerated expansion of the universe. This induces the problem of correct theory of gravity and thus numbers of modified theories of gravity are constructed using modified Einstein-Hilbert actions. Brans-Dicke (BD) theory is one of the most explored examples among various modified theories that provides convenient evidences of various cosmic problems like inflation, early and late behavior of the universe, coincidence problem and cosmic acceleration [2]. This is a generalized form of general relativity (GR) which is constructed by the coupling of scalar field ϕ and tensor field R. It contains a constant coupling parameter ω_{BD} (tuneable parameter) which can be adjusted according to suitable observations. This theory is compatible with Mach's principle, weak equivalence principle and Dirac's large number hypothesis [3]. It is also consistent with solar system observations and experiments (weak field regimes test) for $|\omega| \geq 40,000$ [4].

Gravitational collapse is a process in which stable stellar objects turn into unstable ones under the effects of their own gravity. The formation and dynamics of large scale structures such as stars, celestial cluster and galaxies are investigated through this phenomenon. It is believed that different instability ranges for astronomical bodies lead to different structure formation of collapsing models. Chandrasekhar [5] was the first who explored stability ranges of a spherically symmetric isotropic fluid in GR. He used equation of state involving adiabatic index (Γ) and concluded that the fluid remains unstable for $\Gamma < \frac{4}{3}$. Later on, many researchers [6, 7] investigated dynamical instability of different types of fluids (anisotropic fluid, adiabatic, non adiabatic as well as shearing viscous fluid) in spherical as well as cylindrical configurations and found that stability ranges depend on physical properties of the respective fluid.

It is believed that the study of collapse phenomenon in modified theories may reveal modification hidden in the formation of astronomical structures [8]. In 1969, Nutku [9] explored instability ranges of spherically symmetric

isotropic fluid in BD theory and concluded that BD fluid remains unstable for $\Gamma > \frac{4}{3}$. Kwon et al. [10] discussed instability analysis of the Schwarzschild black hole in BD gravity. Sharif and Kauser [11] investigated stability ranges for spherical as well as cylindrical collapsing models in $f(R)$ theory and found that instability ranges depend upon characteristics of fluids and dark energy components. Sharif and Yousaf [12] studied the effects of electromagnetic field on instability ranges for various models of $f(R)$ gravity. Sharif and Rani [13] explored dynamical instability of spherically symmetric fluid in $f(T)$ theory and concluded that modified terms control instability ranges. In a recent paper [14], we have discussed collapse of spherically symmetric anisotropic BD fluid through instability analysis and found that $0 < \Gamma < 1$ always leads to unstable configuration while $\Gamma > 1$ provides instability only for one particular case.

The behavior of filamentary structures has important implications for the formation of structure in the universe. Galaxy filaments are the largest known cosmic structures in the universe. The filamentary structure is always present in the interstellar medium and instabilities within these filaments create dense medium (dense core) where stars form [15]. N-body simulations of formation of large scale structure describe a wide range of filaments (with a cluster of galaxies forming at the intersection of filaments) [16]. Filamentary structures are associated with cosmic web on the large scales and are used to describe tidal tails (thrown off by merging galaxies) on the small scales [17]. In order to understand fragmentation of filament structures, cylindrically symmetric filament models are widely studied [18].

In this paper, we investigate dynamical instability of cylindrically symmetric filaments collapsing structure in BD gravity. The paper is organized in the following format. The next section discusses BD equations, Darmois junction conditions and dynamical equations. In section **3**, we use perturbation technique to construct hydrostatic equilibrium (collapse equation) and describe instability ranges (at Newtonian and post-Newtonian (pN) limits) for isotropic as well as an anisotropic fluid distributions. Finally, the last section summarizes the results.

2 Brans-Dicke Theory and Dynamical Equations

The BD theory (with self-interacting potential $V(\phi)$) has the following action [3]

$$S=\int d^4x\sqrt{-g}[\phi R-\frac{\omega_{BD}}{\phi}\nabla^{\mu}\phi\nabla_{\mu}\phi-V(\phi)+L_m], \tag{1}$$

where $8\pi G_0=c=1$ and L_m represents matter distribution. Varying Eq.(1) by $g_{\alpha\beta}$ and ϕ, we obtain the following BD equations

$$G_{\alpha\beta} = \frac{1}{\phi}(T^m_{\alpha\beta}+T^{\phi}_{\alpha\beta}), \tag{2}$$

$$\Box\phi = \frac{T^m}{3+2\omega_{BD}}+\frac{1}{3+2\omega_{BD}}[\phi\frac{dV(\phi)}{d\phi}-2V(\phi)]. \tag{3}$$

Here $G_{\alpha\beta}$ is the Einstein tensor, $T^m_{\alpha\beta}$ is the energy-momentum tensor for matter distribution with T^m as its trace and $\Box$ represents d'Alembertian operator. The energy distribution due to scalar field is given by

$$T^{\phi}_{\alpha\beta}=\phi_{,\alpha;\beta}-g_{\alpha\beta}\Box\phi+\frac{\omega_{BD}}{\phi}[\phi_{,\alpha}\phi_{,\beta}-\frac{1}{2}g_{\alpha\beta}\phi_{,\mu}\phi^{,\mu}]-\frac{V(\phi)}{2}g_{\alpha\beta}. \tag{4}$$

Equation (2) gives the BD field equations and (3) is a wave equation for the evolution of scalar field.

We split 4D geometry into interior and exterior regions by considering a timelike 3D hypersurface $\Sigma^{(e)}$ as an external boundary of the respective cylindrical body [7]. The interior region of a collapsing cylindrical filamentary structure is represented by

$$ds^2_-=A^2(t,r)dt^2-B^2(t,r)dr^2-C^2(t,r)d\phi^2-dz^2, \tag{5}$$

where we consider comoving coordinates inside the hypersurface. In order to preserve cylindrical symmetry, the coordinates satisfy the following constraints

$$-\infty\leq t\leq\infty,\quad 0\leq r<\infty,\quad -\infty<z<\infty,\quad 0\leq\phi\leq 2\pi.$$

In stationary or static region, a scalar field becomes constant and all stationary black holes in BD gravity are identical with GR solutions [19]. Therefore,

for exterior region to $\Sigma^{(e)}$, we take line element of static cylindrical black hole given by

$$ds_+^2 = -\frac{2M}{r}d\nu^2 + 2drd\nu - r^2(d\phi^2 + \gamma^2 dz^2), \tag{6}$$

where $M,\ \nu$ and γ describe the total gravitating mass, retarded time, and arbitrary constant, respectively [20]. The interior region is filled with anisotropic matter distribution represented by

$$T^m_{\alpha\beta} = (\rho + p_r)u_\alpha u_\beta - p_r g_{\alpha\beta} + (p_z - p_r)S_\alpha S_\beta + (p_\phi - p_r)K_\alpha K_\beta, \tag{7}$$

where $\rho,\ p_r,\ p_\phi$ and p_z indicate energy density and principal pressure stresses, respectively. The four velocity u_α, unit four-vectors S_α and K_α are calculated as $u_\alpha = A\delta^0_\alpha,\ S_\alpha = \delta^3_\alpha$ and $K_\alpha = C\delta^2_\alpha$ satisfying $u^\alpha u_\alpha = 1,\ S^\alpha S_\alpha = K^\alpha K_\alpha = -1,\ S^\alpha u_\alpha = K^\alpha u_\alpha = S^\alpha K_\alpha = 0$. For the interior region, the BD equations are given in Appendix **A**.

Junction conditions provide smooth connection between interior and exterior regions over $\Sigma^{(e)}$. We consider Darmois junction conditions to discuss connection between two regions [7] and for this purpose we take C-energy (mass function) [21] given by

$$\tilde{E}(t,r) = m(t,r) = \frac{1}{8}(1 - l^{-2}\nabla^\beta\tilde{r}\nabla_\beta\tilde{r}). \tag{8}$$

Here $\tilde{E}(t,r)$ is the gravitational energy per unit specific length of the cylinder, $\tilde{r}$ represents the areal radius, μ shows the circumference radius and l indicates specific length. These are given as follows

$$\tilde{r} = \mu l, \quad \mu^2 = \xi_{(1)\beta}\xi^\beta_{(1)}, \quad l^2 = \xi_{(2)\beta}\xi^\beta_{(2)},$$

where $\xi_{(1)} = \frac{\partial}{\partial\theta}$ and $\xi_{(2)} = \frac{\partial}{\partial z}$ are the respective Killing vectors. For the interior spacetime, Eq.(8) takes the form

$$m(t,r) = \frac{l}{8}\left(1 + \frac{\dot{C}^2}{A^2} - \frac{C'^2}{B^2}\right), \tag{9}$$

where dot and prime show derivatives with respect to t and r, respectively. Since in BD gravity, scalar field and metric tensor are indicated as gravitational variables, therefore $\phi = \phi_{\Sigma^{(e)}} = constant$ at the hypersurface $\Sigma^{(e)}$.

The continuity of first and second fundamental forms (Darmois conditions) yield the following relations

$$r = r_{\Sigma^{(e)}} = constant, \quad m(t,r) - M \overset{\Sigma^{(e)}}{=} \frac{l}{8}, \quad l \overset{\Sigma^{(e)}}{=} 4C,$$
$$\frac{p_r}{\phi} \overset{\Sigma^{(e)}}{=} \frac{-T^{\phi}_{11}}{B^2} - \frac{T^{\phi}_{01}}{AB} = -\frac{V(\phi)}{2\phi}. \tag{10}$$

Dynamical equations obtained from the contracted Bianchi identities describe the conservation of total energy of the system given by

$$\left(\frac{T^{\alpha\beta}_{m}}{\phi} + \frac{T^{\alpha\beta}_{\phi}}{\phi}\right)_{;\alpha} u_\beta = 0, \quad \left(\frac{T^{\alpha\beta}_{m}}{\phi} + \frac{T^{\alpha\beta}_{\phi}}{\phi}\right)_{;\alpha} \chi_\beta = 0, \tag{11}$$

where $\chi_\beta = -B\delta^1_\beta$ (unit four-vector) which provides

$$\left[\frac{\dot{\rho}}{A} - \frac{\rho\dot{\phi}}{\phi^2 A} + (\rho + p_r)\frac{\dot{B}}{AB} + (\rho + p_\phi)\frac{\dot{C}}{AC}\right] + K_1 = 0, \tag{12}$$
$$\left[\frac{p'_r}{B} + \frac{\phi' p_r}{\phi^2 B} + (\rho + p_r)\frac{A'}{AB} + (p_r - p_\phi)\frac{C'}{BC}\right] + K_2 = 0, \tag{13}$$

K_1 and K_2 are mentioned in Appendix **A**.

3 Instability Analysis

Here, we use perturbation approach to construct collapse equation which will be used for instability analysis. We assume that initially, the system is in static equilibrium (metric as well as material parts have radial dependence only) and after that all the dynamical variables along with metric functions are perturbed and time dependence appears [7]. The scalar field, scalar potential and metric tensors have the same time dependence, while the density and pressure bear the same time dependence as follows

$$A(t,r) = A_0(r) + \epsilon T(t) a(r), \tag{14}$$
$$B(t,r) = B_0(r) + \epsilon T(t) b(r), \tag{15}$$
$$C(t,r) = C_0(r) + \epsilon T(t) c(r), \tag{16}$$
$$\phi(r,t) = \phi_o(r) + \epsilon T(t) \Phi(r), \tag{17}$$

$$
\begin{aligned}
p_r(t,r) &= p_{r0}(r) + \epsilon \bar{p}_r(t,r), && (18)\\
p_\phi(t,r) &= p_{\phi 0}(r) + \epsilon \bar{p}_\phi(t,r), && (19)\\
\rho(t,r) &= \rho_0(r) + \epsilon \bar{\rho}(t,r), && (20)\\
V(\phi) &= V_0(r) + \epsilon T(t)\bar{V}(r), && (21)
\end{aligned}
$$

where $0 < \epsilon \ll 1$ and the static distribution is expressed by zero subscript. For static and perturbed configurations of the field as well as dynamical equations, we take $C_0 = r$. The static configuration of BD formalism, perturbed form of BD equations and junction condition (10) are given in Appendix **A**.

The perturbed distribution of first Bianchi identity gives

$$
\bar{\rho} = -\left[\frac{(\rho_0 + p_{r0})b}{B_0} + \frac{(\rho_0 + p_{\phi 0})c}{r} + \frac{\Phi\rho_0}{\phi_0} + A_0\phi_0\bar{K}_1\right]T. \tag{22}
$$

The perturbed form of Eq.(13) provides

$$
\bar{p}_r' + (\bar{\rho} + \bar{p}_r)\frac{A_0'}{A_0} + (\bar{p}_r - \bar{p}_\phi)\frac{1}{r} + \frac{\bar{p}_r\phi_0'}{\phi_0 B_0} + \bar{K}_2\phi_0 B_0 = 0, \tag{23}
$$

where $\bar{K}_1$ and $\bar{K}_2$ are given in appendix **A**. Equation (55) along with junction conditions provides

$$
T(t) = c_1 \exp(\gamma_{\Sigma^{(e)}} t) + c_2 \exp(\lambda_{\Sigma^{(e)}} t), \tag{24}
$$

where $\gamma_{\Sigma^{(e)}} = +\sqrt{\frac{v}{u}}$, $\lambda_{\Sigma^{(e)}} = -\sqrt{\frac{v}{u}}$ with

$$
u \overset{\Sigma^{(e)}}{=} \frac{\Phi}{\phi_0 A_0^2} - \frac{2c}{rA_0^2}, \quad v \overset{\Sigma^{(e)}}{=} \frac{\Phi}{\phi_0}\left[\frac{\omega_{BD}}{\phi_0} - \frac{1}{B_0 r}\right]
$$

and $c_1,\ c_2$ indicate arbitrary constants. Equation (24) shows static and non-static distributions leading to stable as well as unstable phases of gravitating system. For instability analysis, we assume that when the instability phase begins, the system was in complete hydrostatic equilibrium, i.e., $(t = -\infty,\ T(-\infty) = 0)$. Using this assumption in Eq.(24), we have $c_2 = 0$ whereas $c_1 = -1$ is chosen arbitrarily. The corresponding result is described by

$$
T(t) = -\exp(\gamma_{\Sigma^{(e)}} t). \tag{25}
$$

For a real instability regime, we assume only positive values of $\frac{v}{u}$.

For the investigation of instability ranges, we use an equation of state involving adiabatic index Γ [22] given by

$$\bar{p}_j=\Gamma\frac{p_{j0}}{\rho_0+p_{j0}}\bar{\rho}. \tag{26}$$

The adiabatic index evaluates variation of principal stresses (pressures) with respect to density and represents rigidity of the gravitating fluid. We consider Γ to be constant throughout the stability analysis of the fluid. Equations (22) and (26) lead to

$$\begin{aligned}\bar{p}_r &= -\Gamma\left[\frac{b}{B_0}p_{r0}+\frac{c}{r}\frac{\rho_0+p_{\phi0}}{\rho_0+p_{r0}}p_{r0}+\frac{p_{r0}}{\rho_0+p_{r0}}\frac{\Phi\rho_0}{\phi_0}+\frac{p_{r0}A_0\phi_0}{\rho_0+p_{r0}}\bar{K}_1\right]T, &(27)\\ \bar{p}_\phi &= -\Gamma\left[\frac{b}{B_0}\frac{\rho_0+p_{r0}}{\rho_0+p_{\phi0}}p_{\phi0}+\frac{cp_{\phi0}}{r}+\frac{p_{\phi0}\Phi\rho_0}{(\rho_0+p_{\phi0})\phi_0}+\frac{p_{\phi0}A_0\phi_0}{\rho_0+p_{\phi0}}\bar{K}_1\right]T. &(28)\end{aligned}$$

Using Eqs.(51), (22), (27) and (28) in (23), we construct a hydrostatic equation given by

$$\begin{aligned}&\Gamma\left[p_{ro}\left[\frac{bT}{B_0}+\frac{(\rho_0+p_{\phi0})}{(\rho_0+p_{r0})}\frac{cT}{r}+\frac{1}{\rho_0+p_{ro}}A_0\phi_0\bar{K}_1T\right]\right]'-\frac{\Gamma}{r}\left[p_{ro}\left[\frac{bT}{B_0}\right.\right.\\ &\left.\left.+\frac{(\rho_0+p_{\phi0})}{(\rho_0+p_{r0})}\frac{cT}{r}+\frac{p_{r0}}{\rho_0+p_{ro}}A_0\phi_0\bar{K}_1T\right]\right]+\frac{p_{\phi0}}{r}\Gamma\left[p_{\phi o}\left[\frac{bT}{B_0}\frac{(\rho_0+p_{r0})}{(\rho_0+p_{\phi0})}\right.\right.\\ &\left.\left.+\frac{cT}{r}+\frac{1}{\rho_0+p_{\phi o}}A_0\phi_0\bar{K}_1T\right]\right]-\Gamma\left[p_{ro}\left[\frac{bT}{B_0}+\frac{(\rho_0+p_{\phi0})}{(\rho_0+p_{r0})}\frac{cT}{r}+\frac{1}{\rho_0+p_{ro}}\right.\right.\\ &\left.\left.\times A_0\phi_0\bar{K}_1T\right]\right]\frac{A_0'}{A_0}-\left[\frac{bT}{B_0}(\rho_0+p_{r0})+(\rho_0+p_{\phi0})\frac{cT}{r}+A_0\phi_0\bar{K}_1T\right]\\ &\times\frac{A_0'}{A_0}-\Gamma\left[p_{ro}\left[\frac{bT}{B_0}+\frac{(\rho_0+p_{\phi0})}{(\rho_0+p_{r0})}\frac{cT}{r}+\frac{1}{\rho_0+p_{ro}}A_0\phi_0\bar{K}_1T\right]\right]\frac{\phi_0'}{\phi_0B_0}\\ &+\phi_0B_0\bar{K}_2=0.\end{aligned} \tag{29}$$

This represents the general form of collapse equation which describes the instability of hydrostatic equilibrium of gravitating filaments in BD gravity.

3.1 Isotropic Fluid

Here, we analyze instability ranges of isotropic fluid in Newtonian and pN limits. In isotropic fluid, all principal stresses are equal ($p_r=p_\phi=p_z$). Using

this condition in Eq.(29) we obtain the corresponding collapse equation

$$\Gamma\left[p_{ro}\left[\frac{bT}{B_0}+\frac{cT}{r}+\frac{1}{\rho_0+p_{ro}}A_0\phi_0\bar{K}_1T\right]\right]'-\Gamma\left[p_{ro}\left[\frac{bT}{B_0}+\frac{cT}{r}\right.\right.$$
$$\left.\left.+\frac{1}{\rho_0+p_{ro}}A_0\phi_0\bar{K}_1T\right]\right]\frac{A_0'}{A_0}-\left[(\rho_0+p_{r0})(\frac{bT}{B_0}+\frac{cT}{r})\right.$$
$$\left.+A_0\phi_0\bar{K}_1T\right]\frac{A_0'}{A_0}-\Gamma\left[p_{ro}\left[\frac{bT}{B_0}+\frac{cT}{r}+\frac{1}{\rho_0+p_{ro}}A_0\phi_0\bar{K}_1T\right]\right]\frac{\phi_0'}{\phi_0B_0}$$
$$+\phi_0B_0\bar{K}_2=0. \tag{30}$$

Newtonian Limit

The Newtonian limit in BD theory leads to the following

$$\rho_0\gg p_{r0},\quad \rho_0\gg p_{\phi0},\quad B_0=1,\quad A_0=1-\frac{m_0}{rc^2},$$
$$\phi_0=constant,\quad V_0=\bar{V}=0. \tag{31}$$

Using these limits along with (25), the collapse condition turns out to be

$$\Gamma\left[(p_{r0}Z_N)_{,1}-\frac{m_0}{r^2c^2}p_{r0}Z_N\right]-\rho_0Z_N\frac{m_0}{r^2c^2}+K_3<0,$$

which gives

$$\Gamma<\frac{\rho_0Z_N\frac{m_0}{r^2c^2}-K_3}{\left[(p_{r0}Z_N)_{,r}-\frac{m_0}{r^2c^2}p_{r0}Z_N\right]}. \tag{32}$$

Here

$$Z_N=\left(b+\frac{c}{r}\right),\quad K_3=-\left[-\frac{a'm_0}{r^2c^2\phi_0}+\frac{\Phi}{\phi_0}(1-\frac{m_0}{rc^2})-2p_{r0}\frac{\Phi}{\phi_0}\right].$$

This shows that the adiabatic index depends on dynamical properties such as density, pressure, scalar field. To preserve difference between configurations of pressure gradient and gravitational forces, we assume $\Gamma>0$. Thus the celestial objects remain unstable until (32) is satisfied which leads to

$$\frac{\rho_0Z_N\frac{m_0}{r^2c^2}-K_3}{\left[(p_{r0}Z_N)_{,r}-\frac{m_0}{r^2c^2}p_{r0}Z_N\right]}>\Gamma>0. \tag{33}$$

This leads to the following possibilities:

1. $\rho_0 Z_N \frac{m_0}{r^2c^2} - K_3 = \left[(p_{r0}Z_N)_{,r} - \frac{m_0}{r^2c^2}p_{r0}Z_N\right]$;

2. $\rho_0 Z_N \frac{m_0}{r^2c^2} - K_3 < \left[(p_{r0}Z_N)_{,r} - \frac{m_0}{r^2c^2}p_{r0}Z_N\right]$;

3. $\rho_0 Z_N \frac{m_0}{r^2c^2} - K_3 > \left[(p_{r0}Z_N)_{,r} - \frac{m_0}{r^2c^2}p_{r0}Z_N\right]$.

The first and second case along with (33) show that the isotropic system becomes unstable for $0 < \Gamma < 1$. The corresponding expressions lead to

$$p_{r0} = Z_N^{-1}\int_{r_0}^{r} Z_N\left((\rho_0 - 1)\frac{m_0}{r^2c^2} - Z_N^{-1}K_3\right)dr', \tag{34}$$

$$p_{r0} < Z_N^{-1}\int_{r_0}^{r} Z_N\left((\rho_0 - 1)\frac{m_0}{r^2c^2} - Z_N^{-1}K_3\right)dr'. \tag{35}$$

These are the constraint expressions for a collapsing cylindrical isotropic filamentary structure with $0 < \Gamma < 1$. In the third case, the denominator is less than its numerator and hence in (33), Γ can be taken greater than 1. The corresponding instability constraint is given by

$$p_{r0} > Z_N^{-1}\int_{r_0}^{r} Z_N\left((\rho_0 - 1)\frac{m_0}{r^2c^2} - Z_N^{-1}K_3\right)dr'. \tag{36}$$

for which $\Gamma > 1$ and isotropic cylindrical system becomes unstable. It is obvious that if the system is unstable for $\Gamma > 1$, then it will also be unstable for $0 < \Gamma < 1$.

Post-Newtonian Limit

The pN regimes are found upto order c^{-4} by taking

$$A_0 = 1 - \frac{m_0}{rc^2} + \frac{m_0^2}{r^2c^4}, \quad B_0 = 1 + \frac{\alpha m_0}{rc^2}, \quad \phi_0 = constant, \quad V_0 = \bar{V} = 0, \tag{37}$$

where

$$\alpha = \frac{1+\omega_{BD}}{2+\omega_{BD}}.$$

Using pN limits along with Eq.(25) in (29), we obtain

$$0 < \Gamma < \frac{[(\rho_0 + p_{r0})X_{pN}]\left(\frac{m_0}{r^2c^2} - 2\frac{m_0}{r^3c^4}\right) - K_5}{\left[[p_{ro}X_{pN}]' - [p_{ro}X_{pN}]\left(\frac{m_0}{r^2c^2} - 2\frac{m_0}{r^3c^4}\right)\right]}, \tag{38}$$

where

$$X_{pN}=\left[(b(1-\frac{\alpha m_0}{rc^2})+\frac{c}{r})+\frac{K_4}{\rho_0+p_{ro}}\right],$$

K_4 and K_5 are given in Appendix **A**. The expression (38) describes condition for instability of a cylindrical filamentary structure in pN limits. Similar to the Newtonian case, the system collapses for $0<\Gamma<1$ with the following constraints

1. $p_{r0}=X_{pN}^{-1}e^{2r(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4})+\int_{r_0}^{r}Y_{pN}dr'}\int_{r_0}^{r}X_{pN}e^{-2r(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4})-\int_{r_0}^{r}Y_{pN}dr'}$ $(\rho_0(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4})+X_{pN}^{-1}\left[\frac{a'}{\phi_0}(1+\frac{2m_0}{rc^2}(1-\alpha))+\frac{m_0^2}{r^2c^4}(1+4\alpha)\right])-X_{pN}^{-1}\gamma^2_{\Sigma^{(e)}}$ $(\frac{m_0}{r^2c^2}-\frac{4m_0^2}{r^3c^4}+\frac{2\alpha m_0^2}{r^4c^4})$,

2. $p_{r0}<X_{pN}^{-1}e^{2r(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4})+\int_{r_0}^{r}Y_{pN}dr'}\int_{r_0}^{r}X_{pN}e^{-2r(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4})-\int_{r_0}^{r}Y_{pN}dr'}$ $(\rho_0(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4})+X_{pN}^{-1}\left[\frac{a'}{\phi_0}(1+\frac{2m_0}{rc^2}(1-\alpha))+\frac{m_0^2}{r^2c^4}(1+4\alpha)\right])-X_{pN}^{-1}\gamma^2_{\Sigma^{(e)}}$ $(\frac{m_0}{r^2c^2}-\frac{4m_0^2}{r^3c^4}+\frac{2\alpha m_0^2}{r^4c^4})$.

In the third case, $\Gamma>1$ leads to unstable configuration with the following constraint

$$p_{r0}>X_{pN}^{-1}e^{2r(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4})+\int_{r_0}^{r}Y_{pN}dr'}\int_{r_0}^{r}X_{pN}e^{-2r(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4})-\int_{r_0}^{r}Y_{pN}dr'}$$
$$(\rho_0(\frac{m_0}{r^2c^2}-\frac{2m_0^2}{r^3c^4})+X_{pN}^{-1}\left[\frac{a'}{\phi_0}(1+\frac{2m_0}{rc^2}(1-\alpha))+\frac{m_0^2}{r^2c^4}(1+4\alpha)\right])$$
$$-X_{pN}^{-1}\gamma^2_{\Sigma^{(e)}}(\frac{m_0}{r^2c^2}-\frac{4m_0^2}{r^3c^4}+\frac{2\alpha m_0^2}{r^4c^4}),$$

where $Y_{pN}=X_{pN}^{-1}\left[\frac{a'}{\phi_0}(1+\frac{2m_0}{rc^2}(1-\alpha))+\frac{m_0^2}{r^2c^4}(1+4\alpha)\right]$. In this case, $0<\Gamma<1$ is also an instability range.

3.2 Anisotropic Fluid

Here, we have $p_{r0}\neq p_{\phi_0}\neq p_z$ and hydrostatic equilibrium is described by Eq.(29).

Newtonian limit

Using Eq.(31) in (29), we obtain condition for unstable anisotropic filaments as

$$0<\Gamma<\frac{\frac{2p_{r0}\Phi}{\phi_0}+\frac{(p_{r0}-p_{\phi 0})}{r}\left[\frac{c}{r}\right]'+\rho_0(Z_N+K_6)+K_7}{\left[[p_{r0}Z_N]'+\frac{(p_{r0}-p_{\phi 0})}{r}Z_N-p_{r0}Z_N\frac{m_0}{r^2c^2}\right]}, \tag{39}$$

where K_6 and K_7 are mentioned in Appendix **A**. Similar to isotropic case, this implies that for

$$\begin{aligned} p_{r0} &\leq r^{-1}Z_N^{-1}e^{(\frac{m_0}{r^2c^2}-\int_{r_0}^{r}\frac{\Phi}{\phi}+\frac{1}{r}\left[\frac{c}{r}\right]'dr')}\left[\int_{r0}^{r}rZ_Ne^{-(\frac{m_0}{r^2c^2}-\int_{r_0}^{r}\frac{\Phi}{\phi}+\frac{1}{r}\left[\frac{c}{r}\right]'dr')}\left[\frac{p_{\phi_0}}{r}\right.\right.\\ &+ \left.\left.\rho_0(1+K_6)+K_7\right]dr'\right], \end{aligned}$$

Γ lies in $(0,1)$ and the system collapses. If

$$\begin{aligned} p_{r0} &> r^{-1}Z_N^{-1}e^{(\frac{m_0}{r^2c^2}-\int_{r_0}^{r}\frac{\Phi}{\phi}+\frac{1}{r}\left[\frac{c}{r}\right]'dr')}\left[\int_{r0}^{r}rZ_Ne^{-(\frac{m_0}{r^2c^2}-\int_{r_0}^{r}\frac{\Phi}{\phi}+\frac{1}{r}\left[\frac{c}{r}\right]'dr')}\left[\frac{p_{\phi_0}}{r}\right.\right.\\ &+ \left.\left.\rho_0(1+K_6)+K_7\right]dr'\right], \end{aligned}$$

the system becomes unstable for $\Gamma>1$.

Post-Newtonian Limit

The collapse condition of anisotropic cylindrical filaments in pN regime is

$$\Gamma<\frac{\frac{p_{r0}}{r}U_{pN}(\frac{m_0}{r^2c^2}-\frac{m_0^2}{r^3c^4})-(p_{r0}+\rho_0)\left[U_{pN}\right]-K_9}{\left[[p_{r0}U_{pN}]'+\frac{p_{r0}}{r}U_{pN}-\frac{p_{\phi 0}}{r}V_{pN}\right]}, \tag{40}$$

where

$$\begin{aligned} U_{pN} &= \left[b(1-\alpha\frac{m_0}{rc^2})+\frac{(p_{\phi 0}+\rho_0)}{(p_{r0}+\rho_0)}\frac{c}{r}+\frac{1}{(p_{r0}+\rho_0)}K_8\right], \\ V_{pN} &= \left[b(1-\alpha\frac{m_0}{rc^2})\frac{(p_{r0}+\rho_0)}{(p_{\phi 0}+\rho_0)}+\frac{c}{r}+\frac{1}{(p_{\phi 0}+\rho_0)}K_8\right]. \end{aligned}$$

The values of K_8 and K_9 are given in Appendix **A**. The system becomes to unstable for instability ranges $0<\Gamma<1$ if

- $p_{r0}rU_{pN}(\frac{m_0}{r^2c^2}-\frac{m_0^2}{r^3c^4})-(p_{r0}+\rho_0)\,[U_{pN}]-K_9=\big[[p_{r0}U_{pN}]'+\frac{p_{r0}}{r}U_{pN}-\frac{p_{\phi0}}{r}V_{pN}\big]$,
- $p_{r0}rU_{pN}(\frac{m_0}{r^2c^2}-\frac{m_0^2}{r^3c^4})-(p_{r0}+\rho_0)\,[U_{pN}]-K_9<\big[[p_{r0}U_{pN}]'+\frac{p_{r0}}{r}U_{pN}-\frac{p_{\phi0}}{r}V_{pN}\big]$,

and becomes unstable for $\Gamma>1$ if

- $p_{r0}rU_{pN}(\frac{m_0}{r^2c^2}-\frac{m_0}{r^3c^4})-(p_{r0}+\rho_0)\,[U_{pN}]-K_9>\big[[p_{r0}U_{pN}]'+\frac{p_{r0}}{r}U_{pN}-\frac{p_{\phi0}}{r}V_{pN}\big]$.

4 Concluding Remarks

The study of structure formation in modified gravity is an important issue. Cosmic filamentary structures with cylindrical symmetry arise in astrophysics both on large (cosmic web) as well as small (tidal tails) scales. The behavior of these structures has an important role in the formation of structures in the universe. In this paper, we have investigated instability ranges of anisotropic cylindrically symmetric collapsing filaments in BD theory. We have used contracted Bianchi identities to obtain two dynamical equations of collapsing filamentary system. By applying perturbation technique on BD as well as dynamical equations, we separate the unperturbed (static) and perturbed (non-static) distributions of all dynamical relations. We have developed hydrostatic equation (collapse equation) through perturbed configuration of second dynamical equation.

The equation of state involving adiabatic index controls the ranges of instability for a collapsing filamentary structure. We have used collapse equation along with equation of state to investigate the instability ranges of both isotropic as well as anisotropic BD fluid at Newtonian and pN limits. It is concluded that in both approximations the adiabatic index depending upon dynamical properties (energy density, pressure, scalar field terms and some constraints) controls the instability ranges. We have constructed constraints on static radial matter pressure under the effects of scalar field. It is found that the cylindrical filamentary structures always remain unstable for $0<\Gamma<1$, while $\Gamma>1$ is the instability range for the special case. We would like to mention here that the instability ranges for spherical as well as cylindrical distributions in GR depend upon $\Gamma<\frac{4}{3}$ and $\Gamma<1$. In $f(R)$ and $f(T)$ theories, physical variables such as density, pressure and respective modified dark terms provide the instability ranges. The instability range of spherically symmetric isotropic BD fluid is $\Gamma>\frac{4}{3}$ while anisotropic spherical BD fluid always remains unstable for $0<\Gamma<1$ and $\Gamma>1$ leads to collapse only for the special case.

Appendix A

The non-zero components of BD equations for interior region are

$$G_{00}=\frac{1}{\phi}(T^m_{00}+T^\phi_{00})=\frac{1}{\phi}\left(\rho A^2+\frac{\omega_{BD}}{2\phi}(\dot{\phi}^2+\frac{A^2\phi'^2}{B^2})\right)$$
$$-\frac{\dot{\phi}}{\phi}\left(\frac{2\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)+\frac{\phi' A^2}{\phi B^2}\left(\frac{B'}{B}+\frac{C'}{C}\right)+\frac{A^2\phi''}{B^2\phi}-\frac{A^2V(\phi)}{2\phi},\qquad(41)$$

$$G_{01}=\frac{1}{\phi}(T^m_{01}+T^\phi_{01})=\frac{\omega_{BD}}{\phi^2}(\dot{\phi}\phi')+\frac{1}{\phi}\left(\dot{\phi}'-\frac{A'\dot{\phi}}{A}-\frac{\dot{B}\phi'}{B}\right),\qquad(42)$$

$$G_{11}=\frac{1}{\phi}(T^m_{11}+T^\phi_{11})=\frac{1}{\phi}\left(p_rB^2+\frac{\omega_{BD}}{2\phi}(\phi'^2+\frac{B^2\dot{\phi}^2}{A^2})\right)+\frac{\dot{B}\ddot{\phi}}{A^2\phi}$$
$$+\frac{B^2\dot{\phi}}{A^2}\left(\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)-\frac{\phi'}{\phi}\left(\frac{A'}{A}+\frac{C'}{C}\right)+\frac{B^2V(\phi)}{2\phi},\qquad(43)$$

$$G_{22}=\frac{1}{\phi}(T^m_{22}+T^\phi_{22})=\frac{1}{\phi}\left(p_\perp C^2+\frac{\omega_{BD}}{2\phi}(\frac{\dot{C}^2\dot{\phi}^2}{A^2}-\frac{C^2\phi'^2}{B^2})\right)+\frac{\ddot{\phi}C^2}{A^2\phi}$$
$$+\frac{C^2\dot{\phi}}{A^2\phi}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)-\frac{C^2\phi'}{B^2\phi}\left(\frac{A'}{A}+\frac{B'}{B}\right)-\frac{C^2\phi''}{B^2\phi}+\frac{C^2V(\phi)}{2\phi},\qquad(44)$$

$$G_{33}=\frac{1}{\phi}(T^m_{33}+T^\phi_{33})=p_z+\frac{\omega_{BD}}{2\phi^2B^2}\left[\frac{\dot{\phi}^2}{A^2}-\frac{\phi'^2}{B^2}\right]+\frac{\ddot{\phi}}{A^2\phi}$$
$$+\frac{\dot{\phi}}{A^2\phi}\left[\frac{\dot{A}}{A}+\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right]-\frac{\phi'}{B^2\phi}\left[\frac{A'}{A}+\frac{B'}{B}+\frac{C'}{C}\right]-\frac{\phi''}{B^2\phi}+\frac{V(\phi)}{2\phi}\qquad(45)$$

and Eq.(3) becomes

$$\dot{\phi}\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{A^2B}-\frac{\dot{C}}{A^2B}\right)+\frac{\ddot{\phi}}{A^2}+\phi'\left(\frac{A'}{AB^2}-\frac{B'}{B^3}-\frac{C'}{CB^2}\right)-\frac{\phi''}{B^2}$$
$$=\frac{1}{2\omega_{BD}+3}\left[(\rho+3p_r+p_\phi+p_z)+\left(\phi\frac{dV}{d\phi}-2V\right)\right].\qquad(46)$$

The scalar terms K_1 and K_2 of Eqs.(12) and (13) are

$$K_1 = \left(T^\phi_{00}\right)_{,t}A^{-1}-\left(T^\phi_{01}\right)_{,r}A^{-1}B^{-2}+\left(\rho A^{-1}+T^\phi_{00}A^{-2}\right)\phi^{-2}\dot{\phi}$$

$$
\begin{aligned}
&+ \quad T^{\phi}_{01}A^{-1}B^{-2}\phi^{2}\phi' - 2T^{\phi}_{01}B^{3}AB' - T^{\phi}_{01}A^{-2}B^{-2}A', \\
K_2 \quad &= \quad T^{\phi}_{11}B^{-1}\phi'\phi^{-2} + \left(\rho + T^{\phi}_{01}\right)A^{-2}B^{-1}\phi^{-2}\dot{\phi} - \left(T^{\phi}_{01}A^{-2}B^{-2}\right)_{,t}B \\
&- \quad \left(T_{11}B^{-2}\right)_{;r}B.
\end{aligned}
$$

The static distribution of BD field equations is

$$
\frac{\rho_0}{\phi_0} + \frac{\omega_{BD}\phi_0'^2}{2B_0^2\phi_0^2} + \frac{B_0'\phi_0'}{B_0^3\phi_0} + \frac{2\phi_0'}{B_0^2 r\phi_0} + \frac{\phi_0''}{B_0^2\phi_0} - \frac{V_0}{2\phi_0} = \frac{1}{B_0^2 r}\frac{B_0'}{B_0}, \tag{47}
$$

$$
\frac{p_{r0}}{\phi_0} + \frac{\omega_{BD}\phi_0'^2}{2B_0^2\phi_0^2} - \frac{A_0'\phi_0'}{A_0B_0^2\phi_0} - \frac{2\phi_0'}{B_0^2 r\phi_0} + \frac{V_0}{2\phi_0} = \frac{1}{B_0^2 r}\frac{A_0'}{A_0}, \tag{48}
$$

$$
\begin{aligned}
&\frac{p_{\phi 0}}{\phi_0} - \frac{\omega_{BD}\phi_0'^2}{2B_0^2\phi_0^2} - \frac{B_0'\phi_0'}{B_0^3\phi_0} - \frac{A_0'\phi_0'}{A_0B_0^2\phi_0} - \frac{\phi_0''}{B_0^2\phi_0} + \frac{V_0}{2\phi_0} \\
&= \frac{1}{B_0^2}\left[\frac{A_0''}{A_0} + \frac{A_0'}{A_0}\frac{B_0'}{B_0}\right],
\end{aligned} \tag{49}
$$

$$
\begin{aligned}
&\frac{p_z}{\phi_0} - \frac{\phi_0'A_0'}{B_0^2A_0\phi_0} - \frac{\phi_0'B_0'}{B_0^3\phi_0} - \frac{\phi_0'}{B_0^2 r\phi_0} - \frac{\phi_0''}{B_0^2} - \frac{\omega_{BD}\phi_0'^2}{2\phi_0^2B_0^2} + \frac{V_0}{2\phi_0} \\
&= \left(\frac{A_0'}{r} + A_0''\right)\frac{1}{A_0B_0^2} - \frac{B_0'}{B^3}\left(\frac{1}{r} + \frac{A_0'}{A_0}\right).
\end{aligned} \tag{50}
$$

The unperturbed wave equation is

$$
\frac{\phi_0'A_0'}{A_0} - \frac{\phi_0'B_0'}{B_0^2} + \frac{\phi_0'}{rB_0} = \frac{-1}{2\omega_{BD}+3}\left[(\rho_0 + 3p_{r0} + p_{\phi 0} + p_{z0}) + (\phi_0V_0 - 2V_0)\right].
$$

The static distribution of Eq.(12) is identically satisfied in static background while (13) turns out to be

$$
\frac{p_{r0}'}{B_0\phi_0} + \frac{\phi_0'p_{r0}}{\phi_0^2B_0^2} + (\rho_0 + p_{r0})\frac{A_{0'}}{A_0B_0\phi_0} + \frac{1}{B_0\phi_0 r}(p_{r0} - p_{\phi_0}) - K_2' = 0, \tag{51}
$$

where

$$
K_2' = T^{\phi}_{11unp}\frac{\phi_0'}{\phi_0B_0^2} - \frac{p_{r0}'}{\phi_0B_0} - (p_{r0}+\rho_0)\frac{A_0'}{A_0B_0\phi_0} - \frac{(p_{r0}+p_{\phi 0})}{B_0 r\phi_0} + 2T^{\phi}_{11unp}\frac{B_0'}{B_0^2} - \frac{(T^{\phi}_{11unp})_{;r}}{B_0}
$$

and the term T^{ϕ}_{11unp} represents unperturbed form of energy tensor due to scalar field.

The static part of Eq.(10) is

$$p_{r0} \overset{\Sigma^{(e)}}{=} \frac{-V_0}{2}. \tag{52}$$

The perturbed form of BD field equations are

$$\begin{aligned}
&-\frac{2T}{B_0^2}\left[\left(\frac{c}{r}\right)'' - \frac{1}{r}\left(\frac{b}{B_0}\right)' - \left(\frac{B_0'}{B_0}\right)\left(\frac{c}{r}\right)'\right] = -\frac{\bar{\rho}}{\phi_0} - \frac{T\rho_0\Phi}{\phi_0^2}\\
&+\frac{T\omega_{BD}\phi_0'^2 b}{\phi_0^2 B_0^3} - \frac{T\omega_{BD}\bar{\Phi}\phi_0'^2}{B_0^3\phi_0^3} + \frac{\omega_{BD}T\bar{\Phi}'\phi_0'}{B_0^2\phi_0^2} + \frac{T\phi_0'}{\phi_0 B_0^2}\left(\frac{c}{r}\right)' + \frac{T\phi_0'}{\phi_0 B_0^2}\left(\frac{\phi_0 b}{B_0}\right)'\\
&-\frac{2Tb\phi_0'}{\phi B_0^3}\frac{1}{r} + \left[\frac{T}{B_0^2 r} + \frac{TB_0'}{B_0^3}\right]\left[\frac{\Phi}{\phi_0}\right]' + \frac{T\Phi''}{\phi_0 B_0^2} - \frac{2Tb\phi_0''}{B_0^3\phi_0} - \frac{T\phi_0'\Phi}{B_0^2\phi_0^2}\\
&-\frac{TV_0\Phi}{2\phi_0^2} - \frac{T\bar{V}}{2\phi_0},
\end{aligned} \tag{53}$$

$$-\frac{c'}{c} + \frac{A_0'}{A_0} + \frac{b}{cB_0} = \frac{\omega_{BD}\dot{T}\Phi'\dot{\phi}}{\phi_0^2} - \frac{\omega_{BD}\dot{T}\Phi\phi_0'}{\phi_0} + \frac{A_0'\dot{T}c}{rA_0} - \frac{\dot{T}c'}{r} + \frac{\dot{T}b}{rB_0}, \tag{54}$$

$$\begin{aligned}
&-\frac{2\ddot{T}c}{rA_0^2} + \frac{T}{B_0^2 r}\left[\left(\frac{a}{A_0}\right)' + \left(r\frac{A_0'}{A_0}\right)\left(\frac{c}{r}\right)'\right] - \frac{2bA_0'}{rA_0B_0^3} = \frac{\bar{p}_r}{\phi_0} - \frac{Tp_{r0}\Phi}{\phi_0^2}\\
&-\frac{T\omega_{BD}}{\phi_0^2}\left[\frac{\phi_0'^2 b}{B_0^3} + \Phi' - \frac{\Phi\phi_0'^2}{\phi_0}\right] - \frac{T\phi_0'}{\phi_0 B_0^2}\left[\left(\frac{a}{A_0}\right)' + \left(\frac{c}{r}\right)'\right]\\
&+\frac{2Tb\phi_0'}{\phi B_0^3}\left[\frac{A_0'}{A_0} - \frac{1}{r} + \frac{V_0}{2B_0}\right] - \left[\frac{T}{B_0^2 r} + \frac{TA_0'}{B_0^2 A_0} - \frac{TB_0'}{B_0^3}\right]\left[\frac{\Phi}{\phi_0}\right]'\\
&-\frac{\dot{T}b\phi_0'}{\phi_0 B_0^2} + \frac{\ddot{T}\Phi}{A_0^2\phi_0} + \frac{TV_0\Phi}{2\phi_0^2} + \frac{T\bar{V}}{2\phi_0},
\end{aligned} \tag{55}$$

$$\begin{aligned}
&-\frac{b\ddot{T}}{A_0^2 B_0} + \frac{T}{A_0 B_0^2}\left[\left(\frac{a}{A_0}\right)'' + \left(\frac{c}{r}\right)'' + \left(\frac{2A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{1}{r}\right)\left(\frac{a}{A_0}\right)'\right.\\
&\left.-\left(\frac{A_0'}{A_0} + \frac{1}{r}\right)\left(\frac{b}{B_0}\right)' + \left(\frac{A_0'}{A_0} - \frac{B_0'}{B_0} + \frac{2}{r}\right)\left(\frac{c}{r}\right)'\right] = -\frac{\bar{p}_\phi}{\phi_0} - \frac{Tp_{\phi 0}\Phi}{\phi_0^2}\\
&-\frac{T\omega_{BD}\phi_0'^2 b}{2\phi_0^2 B_0^3} - \frac{T\omega_{BD}\Phi'}{2\phi_0^2 B_0^2} - \frac{T\phi_0'}{\phi_0 B_0^2}\left[\left(\frac{a}{A_0}\right)' + \frac{T}{r^2\phi_0}\left(\frac{br^2\phi_0'}{B_0}\right)'\right]\\
&+\frac{2Tb\phi_0'}{\phi B_0^3}\frac{A_0'}{A_0} - \left[\frac{TB_0'}{B_0^3} + \frac{TA_0'}{B_0^3 A_0}\right]\left[\frac{\Phi}{\phi_0}\right]' - \frac{T}{B_0^2\phi_0}\left[\frac{\Phi'}{B_0}\right]' - \frac{T\Phi}{\phi_0 B_0}\left[\frac{\phi_0'}{B_0}\right]'\\
&+\frac{2Tc}{r^2}\left[\frac{\phi_0' B_0'}{B_0^3} + \frac{V_0^2}{2}\right] + \frac{2Tb\phi_0''}{B_0^3\phi_0} + \frac{T\phi_0'\Phi}{B_0^2\phi_0^2} + \frac{TV_0\Phi}{2\phi_0^2} + \frac{T\bar{V}}{2\phi_0},
\end{aligned} \tag{56}$$

$$
\begin{aligned}
&-T\left[\frac{2bA_0''}{A_0B_0^3}+\frac{A_0''}{A_0^2B_0^2}+\frac{a''}{A_0B_0^2}+\frac{A_0'}{B_0^3A_0}\left(\frac{b}{B_0}\right)'-\frac{2bA_0'B_0'}{A_0B_0^3}+\frac{1}{B_0^2}\left(\frac{a}{A_0}\right)'\right.\\
&-\frac{B_0'}{B_0^3}\left(\frac{c}{r}\right)'+\frac{2bB_0}{rB_0^3}-\frac{1}{r}\left(\frac{b}{B_0}\right)'+\frac{2bA_0'}{A_0rB_0^3}+\frac{A_0'}{A_0}\left(\frac{c}{r}\right)'+\frac{1}{rB_0^2}\left(\frac{a}{A_0}\right)'\\
&\left.+\frac{2bA_0'}{A_0B_0^3r}+\frac{A_0'}{A_0B_0^2}\left(\frac{c}{r}\right)'+\frac{1}{rB_0^2}\left(\frac{a}{A_0}\right)'\right]-\frac{b\ddot{T}}{A_0^2B_0}-\frac{c\ddot{T}}{rA_0^2}\\
&=\bar{p}_z-\frac{T}{\phi_0}\left[\frac{b\phi_0'}{B_0}\right]'-\frac{T\phi_0'}{\phi_0B_0^2}\left[\frac{a}{A_0}\right]'-\frac{TA_0'}{B_0^2A_0}\left[\frac{\Phi}{\phi_0}\right]'-\frac{TB_0'}{\phi_0B_0^3}\left[\frac{\Phi}{\phi_0}\right]'\\
&+\frac{2Tb\phi_0'}{B_0^3\phi_0}\left[\frac{A_0'}{A_0}+\frac{1}{r}\right]-\frac{\omega_{BD}T}{\phi_0^2B_0^2}\left[\frac{\Phi'\phi_0'}{\phi_0^2}-\frac{\Phi\phi_0'^2}{\phi_0^3}-\frac{\omega_{BD}b\phi_0'^2}{B_0^3\phi_0^2}\right]\\
&+\frac{\Phi''}{B_0\phi_0}-\frac{T\phi_0''\Phi}{B_0^2\phi_0^2}-\frac{TV_0\Phi}{\phi_0^2},
\end{aligned}
\tag{57}
$$

and perturbed wave equation is given by

$$
\begin{aligned}
&\frac{\phi_0'}{B_0}\left[\frac{TaA_0'}{A_0}-Ta'+\frac{TbB_0'}{B_0}-Tb'+\frac{Tc}{r}-Tc'\right]+2\frac{Tb\phi_0''}{B_0}-\frac{T\Phi''}{B_0^2}\\
&=\frac{1}{2\omega_{BD}+3}\times\left[\bar{\rho}+3\bar{p}_r+\bar{p}_\phi+\bar{p}_z+T\Phi V_0-2\bar{V}\right].
\end{aligned}
\tag{58}
$$

The perturbed configuration of Eq.(10) is

$$
-\bar{p}_r \overset{\Sigma^{(e)}}{=} -\frac{T\Phi p_{r0}}{\phi_0}-\frac{T\Phi V_0}{2\phi_0}+\frac{T\bar{V}}{2\phi_0}. \tag{59}
$$

The perturbed terms $\bar{K}_1$ and $\bar{K}_2$ in Eqs.(22) and (23) are described as

$$
\begin{aligned}
\bar{K_1} &= \dot{T}\left[\frac{\rho_0\Phi}{A_0\phi_0^2}+T^{\phi}_{00(p)}A_0^{-1}+\left(T^{\phi}_{01(p)}\right)'A^{-1}B^{-2}-T^{\phi}_{01(p)}A_0'A^{-2}B^{-2}\right.\\
&- \left.T^{\phi}_{01(p)}B_0'A^{-1}B^{-3}\right],\\
\bar{K_2} &= -T^{\phi}_{11p}\frac{\phi_0'}{\phi_0^2B_0^2}+\left[-2T\Phi\phi_0+T\Phi'-\frac{2Tb\phi_0'}{\phi_0}\right]\frac{T^{\phi}_{11unp}}{\phi_0^2B_0^2}-\frac{\bar{p}_r'}{\phi_0B_0}\\
&+ \frac{p_{r0}'T}{B_0}\left[\frac{b}{B_0}+\frac{\Phi}{\phi_0}\right]+\frac{(\bar{p}_r-\bar{p}_\phi)}{rB_0\phi_0}+\left[\frac{b\Phi}{B_0^2r\phi_0^2}+\frac{1}{rB_0\phi_0^2}(\frac{c}{r})'\right]\\
&\times T(p_{r0}-p_{\phi0})-\frac{(T^{\phi}_{10})_{,t}}{A_0B_0^2}+\frac{4bB_0'}{B_0^3}T^{\phi}_{11unp}-2T^2_{11unp}\frac{Tb'}{B_0^2}+T^{\phi}_{11unp}\frac{Tb}{B_0^2}
\end{aligned}
$$

$$
\begin{aligned}
- & \frac{2T^{\phi}_{11unp}B'_0}{B_0^2}-\frac{(T^{\phi}_{11p})'}{B_0^2}+\left[\frac{b}{B_0}+\frac{\Phi}{\phi_0}+\frac{1}{B_0\phi_0}\left[\frac{a}{A_0}\right]'\right]\frac{(p_{r0}+\rho_0)A'_0}{A_0B_0\phi_0}\\
+ & (\bar{\rho}+\bar{p}_r)\frac{A'_0}{A_0B_0\phi_o}+\frac{T_{11p}B'_0}{B_0^2}.
\end{aligned}
$$

Here $T^{\phi}_{\mu\nu(unp)}$ and $T^{\phi}_{\mu\nu(p)}$ indicate unperturbed as well as perturbed distributions of BD energy part, respectively.

The values of K_4 and K_5 in (38) are given by

$$
\begin{aligned}
K_4 &= \rho_0\frac{\Phi}{\phi_0}+\Phi\frac{m_0}{r^3c^2}+\Phi'(\frac{m_0}{r^2c^2}+(3+2\alpha)\frac{m_0^2}{r^3c^4})+\Phi\left[1+\frac{2m_0}{rc^2}\right.\\
&+ \left.\frac{m_0^2}{r^2c^4}-\frac{2\alpha m_0}{rc^2}+(4\alpha+1)\frac{m_0^2}{r^2c^4}\right],\\
K_5 &= -p_{r0}\frac{\Phi}{\phi_0}(1-\frac{2\alpha}{r^2c^2})+(\rho_0+p_{r0})\left[\frac{a'}{\phi_0}(1+\frac{2m_0}{rc^2}(1-\alpha))+\frac{m_0^2}{r^2c^4}(1+4\alpha)\right]\\
&- \gamma^2_{\Sigma^{(e)}}(\frac{m_0}{r^2c^2}-\frac{4m_0^2}{r^3c^4}+\frac{2\alpha m_0^2}{r^4c^4}).
\end{aligned}
$$

The scalar field terms of (39) are

$$
\begin{aligned}
K_6 &= \gamma_{\Sigma^{(e)}}\left[\rho_0\Phi+(1+\frac{\Phi'}{\phi_0})+m_0(\frac{\Phi\phi_0}{r^3c^2})\right]+\frac{\phi_0}{r}(1-\frac{m_0}{rc^2}),\\
K_7 &= \gamma^2_{\Sigma^{(e)}}\frac{\Phi m_0^2\phi_0}{r^2c^2}-\frac{\rho_0 a' m_0 r}{c^2}.
\end{aligned}
\tag{60}
$$

The values of K_8 and K_9 in (40) are described as

$$
\begin{aligned}
K_8 &= \gamma_{\Sigma^{(e)}}\rho_0\Phi+[\frac{c}{r}]'(1+\frac{m_0^2}{r^2c^4}-\frac{2\alpha m_0}{rc^4})+(1+\frac{m_0^2}{r^2c^4}-\frac{3\alpha m_0}{rc^2})\frac{\Phi'}{\phi_0}\\
&+ \frac{\phi_0}{r}\left(1-\frac{m_0}{rc^2}-\frac{2\alpha m_0}{r^2c^2}\right)+\Phi\phi_0\left(\frac{m_0^2}{r^2c^2}-2\frac{m_0^2}{r^3c^4}\right)-\Phi\phi_0\frac{m_0^2}{r^2c^4}-2\Phi\alpha\frac{m_0^2}{r^2c^4},\\
K_9 &= 1+\frac{\Phi'}{\phi_0}+\frac{\phi_0}{r}\left(1-\frac{m_0}{rc^2}\right)+\phi_0\frac{m_0}{r^3c^3}+\gamma^2_{\Sigma^{(e)}}\frac{\Phi\phi_0 m_0^2}{r^2c^2}.
\end{aligned}
$$

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